

Time-Changed Bessel Processes and Credit Risk*

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Abstract

The Constant Elasticity of Variance (CEV) model is mathematically presented and then used in a Credit-Equity hybrid framework. Next, we propose extensions to the CEV model with default: firstly by adding a stochastic volatility diffusion uncorrelated from the stock price process, then by more generally time changing Bessel processes and finally by correlating stochastic volatility moves to the stock ones. Properties about strict local and true martingales in this study are discussed. Analytical formulas are provided and Fourier and Laplace transform techniques can then be used to compute option prices and probabilities of default.

1 Introduction

It has been widely recognized for at least a decade that the option pricing theory of Black and Scholes (1973) and Merton (1973) is not consistent with market option prices and underlying dynamics. It has been noted that options with different strikes and maturities have different implied volatilities. Indeed, markets take into account in option prices the presence of skewness and kurtosis in the probability distributions of log returns. In order to deal with those effects, one could use stochastic volatility models (e.g. Heston (1993), Hull and White (1987) or Scott (1987)). Another common alternative is to use a deterministic time and stock price dependent volatility function, the so-called local volatility to capture these effects. One would then build the volatility surface by excerpting the values of this function from option prices, thanks to the well-known Derman and Kani (1994) and Dupire (1994) formula.

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One of the first models developed after Black Merton Scholes (1973) is the Constant Elasticity of Variance model pioneered by Cox (1975) where the volatility is a deterministic function of the spot level; This latter model is somehow an ancestor of local volatility models. It has very interesting features since it suggests that common stock returns are heteroscedastic and that volatilities implied by the Black and Scholes formula are not constant, in other words skew exists in this model. Another interesting property is that it takes into account the so called "Leverage Effect" which considers the effects of financial leverage on the variance of a stock: a stock price increase reduces the debt-equity ratio of a firm and therefore decreases the variance of the stock's returns (see for instance Black (1976), Christie (1982) or Schwert (1989)). A last but not least feature of this model is that it has a non-zero probability of hitting 0 and this could be of importance when one is interested in modelling default by defining bankruptcy as the stock price falling to 0.

For the last few years, the credit derivatives market has become more and more important and the issue of modeling default has grown, giving birth to two main classes of models. The first class is the structural models of the firm pioneered by Merton (1974) where bankruptcy occurs if the asset value falls to a boundary determined by outstanding liabilities. Other early work on such models was done by Black and Cox (1976) and Geske (1977). The other class commonly called reduced-form models is less ambitious than structural models. They consider the time of default as an exogenous parameter that they calibrate under a risk neutral probability to market data. These models were developed by Artzner and Delbaen (1995), Jarrow and Turnbull (1995), Duffie, Schroder and Skiadas (1996) and Madan and Unal (1998).

The credit risk is also a component of the equity derivatives market as it may appear in convertible bonds or more generally in Capital Structure Arbitrage for people that embedded it from out-of-the money puts. It is then clear that having a consistent modeling of equity and credit is essential to eventually be able to manage those cross-asset positions. Indeed, a market standard has been developed during the last few years which involves a jump diffusion dynamics for the stock price with a local probability of default for the jump factor. This kind of model has been presented for instance in Ayache, Forsyth and Vetzal (2003). An important drawback of this modeling is that the stock has to jump to zero in order to default, which isn't a realistic assumption as we can see on several historical data and as argued in Atlan and Leblanc (2005).

The necessity to have stock price diffusions that don't jump to zero in order to default and still have a non-zero probability of falling to zero leads us to naturally consider CEV processes. Moreover, CEV models have the advantage to provide closed-form formulas for European vanilla options and for the probability of default. Those computations were originally performed by Cox (1975) in the case where the stock can default and by Emanuel and McBeth (1982) when the stock never defaults. Then, one may want to add a stochastic volatility process to the CEV diffusion in order to capture some volatility features such as a smile or such as a more realistic volatility term structure. Finally, to get more dependency between the stock price and the volatility, one may add some

correlation.

Those guidelines lead us to study in section 2 the one-dimensional marginals, the first-passage times below boundaries and the default of martingality of Constant Elasticity of Variance processes, mainly by relating those latest to Bessel processes. In section 3, we propose a CEV model that is stopped at its default time and we provide closed form formulas for European vanilla options, Credit Default Swaps and Equity Default Swaps. Section 4 extends the Constant Elasticity of Variance framework to a Constant Elasticity of Stochastic Variance one by firstly adding a stochastic volatility to the CEV diffusion and in a second time more generally consider time-changed Bessel processes with a stochastic integrated time change. Quasi-analytical formulas conditionally on the knowledge of the law of the time change are provided for vanilla options and CDSs and examples are given. Section 5 adds a correlation term to the general time-changed power of Bessel process framework, once again quasi analytical formulas conditionally on the knowledge of the joint law of the time change and a process related to the rate of time change are provided for probabilities of default and for vanilla options, and computations for several examples are shown. All the models proposed in this paper are true martingales and the martingale property is carefully proven for the different frameworks. Finally, section 6 concludes and presents possible extensions of this work.

Convention *For strictly negative dimensions we define squared Bessel processes up to their first hitting time of 0 after which they remain at 0.*

We set this convention because we wish to consider positive Bessel processes. For a study of negative dimension Bessel processes with negative values, we refer to Göing-Jaeschke and Yor (2003).

2 A Mathematical Study of CEV Processes

2.1 Space and Time Transformations

A reason why Bessel processes play a large role in financial mathematics is that they are closely related to widely used models such as Cox, Ingersoll and Ross (1985), i.e. the CIR family of diffusions for interest rates framework, such as the Heston (1993) stochastic volatility model or even to the Constant Elasticity of Variance model of Cox (1976). They are more generally related to exponential of time-changed Brownian motions thanks to Lamperti (1972) representations.

Let us now concentrate on the CIR family of diffusions: they solve the following type of stochastic differential equations:

$$dX_t = (a - bX_t)dt + \sigma\sqrt{|X_t|}dW_t \quad (1)$$

with $X_0 = x_0 > 0$, $a \in \mathbb{R}_+$, $b \in \mathbb{R}$, $\sigma > 0$ and W_t a standard Brownian motion. This equation admits a strong (e.g. adapted to the natural filtration of W_t)

unique solution that takes values in \mathbb{R}_+ .

Let us remark that squared Bessel processes of dimension $\delta > 0$ can be seen as a particular case of a CIR process with $a = \delta$, $b = 0$ and $\sigma = 2$. We also recall that a Bessel process R_t solves the following diffusion equation

$$dR_t = dW_t + \frac{\delta - 1}{2R_t} dt$$

where for $\delta = 1$, the latter $\frac{\delta-1}{2R_t} dt$ must be replaced by a local time term.

One is now interested in the representation of a CIR process in terms of a time-space transformation of a Bessel Process:

Lemma 2.1 *A CIR Process X_t which solves equation (1) can be represented in the following form:*

$$X_t = e^{-bt} BESQ_{(\delta, x_0)}\left(\frac{\sigma^2}{4b}(e^{bt} - 1)\right) \quad (2)$$

where $BESQ_{(\delta, x_0)}$ denotes a squared Bessel Process starting from x_0 at time $t = 0$ of dimension $\delta = \frac{4a}{\sigma^2}$

Proof : This lemma results from the identification of two continuous functions f and g (with g strictly increasing and $g(0) = 0$) such as

$$X_t = f(t) BESQ_{(\delta, x_0)}(g(t))$$

To do so, we apply Itô's formula and Dambis (1965), Dubins-Schwarz (1965) theorem ■

This relation is widely used in finance, for instance in Geman and Yor (1993) or Delbaen and Shirakawa (2002).

Let us now introduce the commonly called CEV (Constant Elasticity of Variance), which was introduced by Cox (1975, 1996) and that solves the following equation:

$$dX_t = \mu X_t dt + \sigma X_t^\alpha dW_t \quad (3)$$

with $X_0 = x_0 > 0$, $\alpha \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\sigma > 0$ and W_t a standard brownian motion.

Lemma 2.2 *A CEV Process X_t which solves equation (3) can be represented as a power of a CIR process, indeed for $\beta = 2(\alpha - 1)$, $1/X_t^\beta$ solves*

$$d\left(\frac{1}{X_t^\beta}\right) = \left(a - b\frac{1}{X_t^\beta}\right)dt + \Sigma \sqrt{\left|\frac{1}{X_t^\beta}\right|} dW_t \quad (4)$$

where $a = \frac{\beta(\beta+1)\sigma^2}{2}$, $b = \beta\mu$, $\Sigma = -\beta\sigma$ and .

Proof : This lemma is just an application of Itô's Lemma. ■

As a consequence of Lemma 2.1 and Lemma 2.2, one obtains the following representation for a CEV process:

Proposition 2.3 *A CEV Process X_t solution of equation (3) can be represented in the following form:*

$$X_t = e^{\mu t} BESQ_{\left(\frac{2\alpha-1}{\alpha-1}, x_0^{-2(\alpha-1)}\right)}^{\frac{1}{2(1-\alpha)}} \left(\frac{(\alpha-1)\sigma^2}{2\mu} (e^{2(\alpha-1)\mu t} - 1) \right) \quad (5)$$

where $BESQ_{(\delta, x_0)}$ denotes a squared Bessel Process starting from x_0 at time $t = 0$ of dimension δ .

2.2 Distributions and Boundaries

We will now recall well known results about squared Bessel processes and deduce some properties about CEV processes.

Path Properties

Proposition 2.4 *According to its dimension, the squared Bessel process has different properties:*

- (i) if $\delta \leq 0$, 0 is an absorbing point.
- (ii) if $\delta < 2$, $\{0\}$ is reached a.s.
- (iii) if $\delta \geq 2$, $\{0\}$ is polar.
- (iv) if $\delta \leq 2$, $BESQ$ is recurrent.
- (v) if $\delta \geq 2$, $BESQ$ is transient.
- (vi) if $0 < \delta < 2$, $\{0\}$ is instantaneously reflecting.

Proof : The proof can be found in Revuz and Yor (2001). ■

As a consequence, one may give some properties of the CEV diffusions. A topic of interest for the remaining of the paper is whether or not $\{0\}$ is reached by a CEV process.

Proposition 2.5 *According to the value of α , the CEV diffusion has different properties:*

- (i) if $\alpha < 1$, $\{0\}$ is reached a.s.
- (ii) if $\alpha \leq \frac{1}{2}$, $\{0\}$ is instantaneously reflecting.
- (iii) if $\frac{1}{2} < \alpha < 1$, $\{0\}$ is an absorbing point.
- (iv) if $\alpha > 1$, $\{0\}$ is an unreachable boundary.

Proof : It is a consequence of the previous proposition and of Proposition 2.3. ■

Distributional Properties

It is important to notice that the law of a squared Bessel process can be seen in terms of non-central chi-square density:

Lemma 2.6 *For any $BESQ_{\delta,x}$, one has:*

$$BESQ_{\delta,x}(t) \stackrel{(d)}{=} tV^{(\delta, \frac{\delta}{t})} \quad (6)$$

where $V^{(a,b)}$ is a non-central chi-square r.v. with a degrees of freedom and non-centrality parameter $b \geq 0$. Its density is given by:

$$f_{a,b}(v) = \frac{1}{2^{\frac{a}{2}}} \exp\left(-\frac{1}{2}(b+v)\right) v^{\frac{a}{2}-1} \sum_{n=0}^{\infty} \left(\frac{b}{4}\right)^n \frac{v^n}{n! \Gamma(\frac{a}{2} + n)} \quad (7)$$

Proof : This proof results from simple properties of Laplace transforms and can be found for instance in Delbaen and Shirakawa (2002). ■

We leave to the reader the calculation of the CEV density in terms of non-central chi-square distributions.

Let us recall a useful result for the remaining of the paper on the moments of a squared Bessel process:

Corollary 2.7 *If $V^{(a,b)}$ is a non-central chi-square r.v. with a degrees of freedom and noncentrality parameter $b \geq 0$, then for any real constants c and d :*

$$\mathbb{E}[(V^{(a,b)})^c \mathbf{1}_{\{V^{(a,b)} \geq d\}}] = e^{-\frac{b}{2}} 2^c \sum_{n \geq 0} \left(\frac{b}{2}\right)^n \frac{\Gamma(n + \frac{a}{2} + c)}{n! \Gamma(\frac{a}{2} + n)} G(n + \frac{a}{2} + c, \frac{d}{2}) \quad (8)$$

where G is defined as follows:

$$G(x, y) = \int_{z \geq y} \frac{z^{x-1} e^{-z}}{\Gamma(x)} \mathbf{1}_{\{z > 0\}} dz$$

Proof : This calculation is a simple application of Lemma 2.6. ■

Finally, for the computations involved in this paper, one recalls the two following identities on the complementary non-central chi-square distribution function Q that one can find in Johnson and Kotz (1970):

$$\begin{aligned} Q(2z, 2\nu, 2\kappa) &= \sum_{n \geq 1} g(n, \kappa) G(n + \nu - 1, z) \\ 1 - Q(2\kappa, 2\nu - 2, 2z) &= \sum_{n \geq 1} g(n + \nu - 1, \kappa) G(n, z) \end{aligned}$$

where $g(x, y) = -\frac{\partial G}{\partial y}(x, y)$.

First-Hitting Times

We now concentrate on the first hitting time of 0 by a Bessel process. For this purpose, let us consider a Bessel Process R of index $\nu > 0$ starting from 0 at time 0, then, one has:

$$L_1(R) \stackrel{(d)}{=} \frac{1}{2Z_\nu} \quad (9)$$

where $L_1(R) = \sup\{t > 0, R_t = 1\}$ and Z_ν is a gamma variable with index ν that has the following density:

$$\mathbb{P}(Z_\nu \in dt) = \frac{t^{\nu-1} e^{-t}}{\Gamma(\nu)} \mathbf{1}_{\{t>0\}} dt \quad (10)$$

This result is due to Gettoor (1979). Thanks to results on time reversal (see Williams (1974), Pitman and Yor (1980) and Sharpe (1980)), we have:

$$(\hat{R}_{T_0-u}; u < T_0(\hat{R})) \stackrel{(d)}{=} (R_u; u < L_1(R)) \quad (11)$$

where \hat{R} is a Bessel Process, starting from 1 at time 0 of dimension $\delta = 2(1 - \nu)$ and $T_0(\hat{R}) = \inf\{t > 0, \hat{R}_t = 0\}$. As a consequence, one has:

$$T_0(\hat{R}) \stackrel{(d)}{=} \frac{1}{2Z_\nu} \quad (12)$$

Using the scaling property of the Squared Bessel Process, one may write:

$$T_0(BESQ_x^\delta) \stackrel{(d)}{=} \frac{x}{2Z_\nu} \quad (13)$$

with $\delta = 2(1 - \nu)$.

Hence, we are now able to state the proposition below:

Proposition 2.8 *The probability of a CEV diffusion solution of equation (3) to reach 0 at time T with $\alpha < 1$ is given by:*

$$\mathbb{P}(T_0 \leq T | X_0 = x_0) = G\left(\frac{1}{2(1-\alpha)}, \zeta_T\right) \quad (14)$$

where G and ζ_T are defined as follows:

$$G(x, y) = \int_{z \geq y} \frac{z^{x-1} e^{-z}}{\Gamma(x)} \mathbf{1}_{\{z>0\}} dz \quad (15)$$

$$\zeta_T = \frac{\mu x_0^{2(1-\alpha)}}{(1-\alpha)\sigma^2(1 - e^{2(\alpha-1)\mu T})} \quad (16)$$

Proof : This proof is just a consequence of Proposition 2.3 and equation (13). ■

Remark 2.9 *The calculation of the probability of default was originally done by Cox (1975).*

In order to compute first-passage times of scalar Markovian diffusions below a fixed level, let us recall Itô and McKean (1974) results. If $(X_t, t \geq 0)$ is scalar

Markovian time-homogeneous diffusion starting from x_0 at time 0 of infinitesimal generator \mathcal{L} and that we define $\tau_H = \inf\{t \geq 0; X_t \leq H\}$ for $H < x_0$, then for any $\lambda > 0$, we have

$$\mathbb{E}[e^{-\lambda\tau_H}] = \frac{\phi_\lambda(x_0)}{\phi_\lambda(H)}$$

where ϕ_λ is solution of the ODE

$$\mathcal{L}\phi = \lambda\phi$$

with the following limit conditions:

$$\lim_{x \rightarrow \infty} \phi_\lambda(x) = 0$$

If 0 is a reflecting boundary then $\phi_\lambda(0+) < \infty$

If 0 is an absorbing boundary then $\phi_\lambda(0+) = \infty$

As a first example, let us now consider the first-hitting time below a fixed level $0 < y \leq x$ of a Bessel process R_t of dimension $\delta = 2(\nu + 1)$ starting from x :

$$\tau_y = \inf\{t \geq 0; R_t \leq y\}$$

The law of τ_y (see Itô and McKean (1974), Kent (1978) or Pitman and Yor (1980)) is obtainable from the knowledge of its Laplace transform \mathcal{L} . One has for any positive λ

$$\begin{aligned} \mathcal{L}(\lambda) &= \mathbb{E}[e^{-\lambda\tau_y}] \\ &= \frac{x^{-\nu} K_\nu(x\sqrt{2\lambda})}{y^{-\nu} K_\nu(y\sqrt{2\lambda})} \end{aligned}$$

where $\nu \in \mathbb{R} \setminus \mathbb{Z}$ and K_ν is a Modified Bessel function defined as follows:

$$\begin{aligned} K_\nu(x) &= \frac{\pi}{2 \sin(\nu\pi)} (I_{-\nu}(x) - I_\nu(x)) \\ I_\nu(x) &= \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)} \end{aligned}$$

As a second example that will be useful for the computation of EDS prices, let us write the infinitesimal generator of a CEV process:

$$\mathcal{L}_{CEV}\phi = \sigma^2 x^{2\alpha} \frac{d^2\phi}{dx^2} + \mu x \frac{d\phi}{dx}$$

that must solve

$$\mathcal{L}_{CEV}\phi = \lambda\phi$$

with the following conditions:

ϕ_λ is a decreasing function, $\lim_{x \rightarrow \infty} \phi_\lambda(x) = 0$

If $\alpha \leq \frac{1}{2}$, then $\phi_\lambda(0+) = \infty$
 If $\frac{1}{2} < \alpha < 1$, then $\phi_\lambda(0+) < \infty$

We obtain the following result whose computations of the Laplace transforms were originally performed by Davydov and Linetsky (2001):

Proposition 2.10 *For a CEV process solution of (3) with $\alpha < 1$ and $\mu \neq 0$, then*

$$\phi_\lambda(x) = x^{\alpha-\frac{1}{2}} \exp\left(-\frac{\mu x^{2(1-\alpha)}}{\sigma^2(1-\alpha)}\right) W_{k,m}\left(\frac{|\mu|x^{2(1-\alpha)}}{\sigma^2(1-\alpha)}\right) \quad (17)$$

where

$$k = \operatorname{sgn}(\mu)\left(\frac{1}{4(1-\alpha)} - \frac{1}{2}\right) - \frac{\lambda}{2|\mu|(1-\alpha)} \quad \text{and} \quad m = \frac{1}{4(1-\alpha)}$$

and $W_{k,m}$ is a Whittaker function

The definition of the Whittaker function can be found for instance in Abramowitz and Stegun (1972).

2.3 Loss of Martingality

Let us now state a result on some martingale properties of Bessel processes which play an essential role in pricing theory as is well known:

Theorem 2.11 *Let R_t be a Bessel process of dimension δ starting from $a \neq 0$, then:*

- (i) *If $\delta \leq 0$, $R_t^{2-\delta}$ is a true martingale up to the first hitting time of 0.*
- (ii) *If $0 < \delta < 2$, the process $R_t^{2-\delta} - L_t$ is a martingale where L_t is a continuous increasing process carried by the zeros of $(R_t, t \geq 0)$.*
- (iii) *If $\delta = 2$, $\log(R_t)$ is a strict local martingale.*
- (iv) *If $\delta > 2$, $R_t^{2-\delta}$ is a strict local martingale. Moreover, the default of martingality is*

$$\gamma^{(\delta)}(t) = \mathbb{E}[R_0^{2-\delta}] - \mathbb{E}[R_t^{2-\delta}] = a^{2-\delta} \mathbb{P}_a^{(4-\delta)}(T_0 \leq t) \quad (18)$$

where \mathbb{P}_a^δ is the law of $(R_t^{(\delta)}, t \geq 0)$.

Proof : (i) and (ii): Since $\{0\}$ is reached a.s., we need to apply Itô's formula in a positive neighborhood of 0. Let us consider $\epsilon > 0$. We have:

$$(\epsilon + R_t^2)^{1-\frac{\delta}{2}} = (\epsilon + a^2)^{1-\frac{\delta}{2}} + (2-\delta) \int_0^t (\epsilon + R_s^2)^{-\frac{\delta}{2}} R_s dW_s + \epsilon \delta \left(1 - \frac{\delta}{2}\right) \int_0^t \frac{ds}{(\epsilon + R_s^2)^{\frac{\delta}{2}+1}}$$

Then, as ϵ tends to zero, it is easy to see the first term of the right hand side is a true martingale for $\delta < 2$ and that the second term of the right hand side is increasing whose support is the zeros of $(R_t, t \geq 0)$ when $\delta \geq 0$. If $\delta < 0$, $T_t^{2-\delta}$

is a true martingale.

(iii): By applying Itô formula, we obtain

$$\log(R_t) = \log(R_0) + \int_0^t \frac{dW_s}{R_s}$$

We then see that $\log(R_t)$ is a local martingale. We prove that it is a strict local martingale by first using the fact that

$$\mathbb{P}_a^\delta = \mathbb{P}_0^\delta * \mathbb{P}_a^0$$

then writing that

$$\begin{aligned} \mathbb{E}[\log(R_t)] &= \frac{1}{2} \mathbb{E}[\log(R_t^2)] = \frac{1}{2} \mathbb{E}[\log(BESQ_{2,0}(t) + BESQ_{0,a}(t))] \\ &\geq \frac{1}{2} \mathbb{E}[\log(BESQ_{2,0}(t))] \end{aligned}$$

and finally since $BESQ_{2,0}(t) \stackrel{d}{=} 2t\mathbf{e}$ where \mathbf{e} is a standard exponential, we obtain

$$\mathbb{E}[\log(R_t)] \geq C + \frac{1}{2} \log(t) \longrightarrow_{t \rightarrow \infty} +\infty$$

which shows that $\log(R_t)$ is not a true martingale.

(iv): To compute $\gamma^{(\delta)}$, we will need the following result:

Lemma 2.12 *Let $(R_t^{(\delta)}, t \geq 0)$ be a Bessel process of dimension $\delta > 2$ starting from $a \neq 0$, then*

$$\mathbb{P}_{a|\mathcal{R}_t \cap \{t < T_0\}}^{4-\delta} = \left(\frac{R_t^{(\delta)}}{a} \right)^{2-\delta} \cdot \mathbb{P}_{a|\mathcal{R}_t}^\delta$$

where \mathcal{R}_t is the canonical filtration of the Bessel process and T_0 the first-hitting time of the level 0.

Proof : This property results from a double application of Girsanov Theorem by computing

$$\frac{d\mathbb{P}_{a|\mathcal{R}_t \cap \{t < T_0\}}^{4-\delta}}{d\mathbb{P}_{a|\mathcal{R}_t}^2} \quad \text{and} \quad \frac{d\mathbb{P}_{a|\mathcal{R}_t}^\delta}{d\mathbb{P}_{a|\mathcal{R}_t}^2}$$

Then, by identification, one gets the announced result. A more general result can be found in Yor (1992). ■

We may then write

$$\mathbb{E}^{(\delta)}[R_t^{2-\delta}] = \mathbb{E}^{(4-\delta)}[a^{2-\delta} \mathbf{1}_{\{t < T_0\}}]$$

and consequently compute the default of martingality. ■

A proof in the case $0 < \delta < 2$ can be found in Donati-Martin et al. (2006) and proofs when $\delta > 2$ exist in Elworthy, Li and Yor (1999). As a consequence, we obtain similar results for a CEV process.

Proposition 2.13 *Let X_t be a CEV Process of elasticity α solving the following equation*

$$dX_t = \mu dt + \sigma X_t^\alpha dW_t$$

then:

- (i) *If $\alpha \leq \frac{1}{2}$, the process $e^{-\mu t} X_t$ is a true martingale up to the first hitting time of 0.*
- (ii) *If $\frac{1}{2} < \alpha < 1$, the process $e^{-\mu t} X_t - L_t^X$ is a martingale where L_t^X is a continuous increasing process carried by the zeros of $(X_t, t \geq 0)$ and consequently $e^{-\mu t} X_t$ is a true martingale up to the first hitting time of 0.*
- (ii) *If $\alpha = 1$, $e^{-\mu t} X_t$ is a geometric Brownian motion and hence a martingale.*
- (iii) *If $\alpha > 1$, $e^{-\mu t} X_t$ is a strictly local martingale. Moreover, the default of martingality is*

$$\gamma_X(t) = \mathbb{E}[X_0] - \mathbb{E}[e^{-\mu t} X_t] = x_0 G\left(\frac{1}{2(\alpha-1)}, \zeta_T\right) \quad (19)$$

where G and ζ_T are defined as follows:

$$\begin{aligned} G(x, y) &= \int_{z \geq y} \frac{z^{x-1} e^{-z}}{\Gamma(x)} \mathbf{1}_{\{z > 0\}} dz \\ \zeta_T &= \frac{\mu x_0^{2(1-\alpha)}}{(\alpha-1)\sigma^2(e^{2(\alpha-1)\mu T} - 1)} \end{aligned}$$

Proof : This is just an application of Theorem 2.11, Proposition 2.3 and equation (14). ■

A proof of the failure of the martingale property can be found in Lewis (1998).

Remark 2.14 *For $\alpha > 1$, one has $\forall(t, K) \in \mathbb{R}_+^2$:*

$$\mathbb{E}[(e^{-\mu t} X_t - K)_+] - \mathbb{E}[(K - e^{-\mu t} X_t)_+] + \gamma_X(t) = \mathbb{E}[X_0] - K \quad (20)$$

The last equation shows that in the case of a strictly local martingale, the Call price must incorporate the default of martingality in order to remain in a No Arbitrage model. For a study on option pricing for strict local martingales, we refer to Madan and Yor (2006) for continuous processes and to Chybyryakov (2006) for jump-diffusion processes. Lewis (2000) also did this study in the case of explosions with stochastic volatility models and in particular for a CEV diffusion.

3 Credit-Equity Modelling

3.1 Model Implementation

Usually, in the mathematical finance literature, one defines a CEV diffusion for the stock price dynamics S to be

$$\frac{dS_t}{S_t} = \mu dt + \sigma S_t^{\alpha-1} dW_t$$

First of all, in a credit perspective, we will just consider the case $\alpha < 1$ since we are interested in models with a non-zero probability of default. Once the stock has reached zero, the firm has bankrupted and that is the reason why we stop the CEV diffusion at its first default time. Then from what has been proven above, we know that the stock price process hence defined is a true martingale and that ensures the Absence of Arbitrage and moreover the uniqueness of the solution. Hence, the stock price diffusion now becomes under the risk-neutral pricing measure:

$$\frac{dS_t}{S_t} = rdt + \sigma S_t^{\alpha-1} dW_t \quad \text{if} \quad t < \tau.$$

$$S_t = 0 \quad \text{if} \quad t \geq \tau.$$

where $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$. In other words, the stock price process considered is nothing else than a stopped CEV diffusion $(S_{t \wedge \tau})_{t \geq 0}$.

Remark 3.1 *Delbaen and Shirakawa (2002) showed the existence of a risk-neutral probability measure whose uniqueness is only ensured on the stock price filtration considered at time τ $\mathcal{F}_\tau = \sigma(S_t, t \leq \tau)$. Since our purpose is to compute the price of options whose payoffs are \mathcal{F}_τ -measurable, we have the uniqueness of the no-arbitrage probability.*

3.2 European Vanilla Option Pricing

Lemma 2.12 states that

$$\mathbb{P}_{x|\mathcal{R}_t \cap \{t < T_0\}}^{4-\delta} = \left(\frac{R_t^{(\delta)}}{x} \right)^{2-\delta} \cdot \mathbb{P}_{x|\mathcal{R}_t}^\delta \quad (21)$$

Thanks to this identity, we obtain the law of the stopped CEV diffusion at a given time. Lemma 2.6 and Corollary 2.7 enable us to compute the call and put option price:

For the call C_0 option price

$$\begin{aligned} C_0 &= e^{-rT} \mathbb{E}[(S_{T \wedge \tau} - K)_+] \\ &= e^{-rT} \mathbb{E}[(S_T - K)_+ \mathbf{1}_{T < \tau}] \end{aligned}$$

and the put P_0 option price:

$$\begin{aligned} P_0 &= e^{-rT} \mathbb{E}[(K - S_{T \wedge \tau})_+] \\ &= e^{-rT} \mathbb{E}[(K - S_T)_+ \mathbf{1}_{T < \tau}] + K e^{-rT} \mathbb{P}(\tau \leq T) \end{aligned}$$

Consequently, for the call price:

$$C_0 = S_0 Q(z_T, 2 + \frac{1}{1-\alpha}, 2\zeta_T) - K e^{-rT} (1 - Q(2\zeta_T, \frac{1}{1-\alpha}, z_T))$$

and for the put price:

$$\begin{aligned}
P_0 &= Ke^{-rT} \left(Q\left(2\zeta_T, \frac{1}{1-\alpha}, z_T\right) - G\left(\frac{1}{2(1-\alpha)}, \zeta_T\right) \right) \\
&\quad - S_0 \left(1 - Q\left(z_T, 2 + \frac{1}{1-\alpha}, 2\zeta_T\right)\right) + Ke^{-rT} \mathbb{P}(\tau \leq T) \\
&= Ke^{-rT} Q\left(2\zeta_T, \frac{1}{1-\alpha}, z_T\right) - S_0 \left(1 - Q\left(z_T, 2 + \frac{1}{1-\alpha}, 2\zeta_T\right)\right)
\end{aligned}$$

where

$$\begin{aligned}
z_T &= \frac{2rK^{2(1-\alpha)}}{\sigma^2(1-\alpha)(e^{2(1-\alpha)rT} - 1)} \\
\zeta_T &= \frac{rS_0^{2(1-\alpha)}}{(1-\alpha)\sigma^2(1 - e^{-2(1-\alpha)rT})}
\end{aligned}$$

Hence, one easily verifies that the put-call parity is satisfied. Closed-form CEV option pricing formulas were originally computed by Cox (1975) for $\alpha < 1$ and Schroder (1989) expressed those formulas in terms of non-central chi-square distributions. Computing option prices using the squared Bessel processes distributions was done by Delbaen and Shirakawa (2002).

3.3 Pricing of Credit and Equity Default Swaps

Since we are dealing with default probabilities, it is obvious to consider derivative products relying on these probabilities. One of the most liquid protection instruments against default is the Credit Default Swap (CDS). The buyer of the protection agrees to pay periodical amounts until a default time (if it occurs) and in exchange receives a cash amount which is a notional amount minus a recovery rate in the case the company on which the contract is written, defaults. The payoff of such kind of contract is:

$$\Pi_{CDS} = - \sum_{i=1}^n e^{-rT_i} C \mathbf{1}_{\{\tau > T_i\}} + e^{-r\tau} (1 - R) \mathbf{1}_{\{\tau \leq T_n\}}$$

where C is the periodical coupon, T_1, \dots, T_n the payment dates, R the recovery rate assumed to be deterministic and τ the default time. For simplicity purposes, we consider in this paper deterministic interest rates. The CDS Fair Price is the expectation of the payoff conditionally to the spot price filtration taken at the pricing time, e.g.:

$$CDS_t(T_1, T_n; C; R) = -C \sum_{i=1}^n e^{-r(T_i-t)} \mathbb{P}(\tau > T_i | S_t) + (1-R) \mathbb{E}[e^{-r(\tau-t)} \mathbf{1}_{\{\tau \leq T_n\}} | S_t]$$

By absence of arbitrage, one must have $CDS_t(T_1, T_n; C; R) = 0$ and then

$$C = \frac{(1-R) \mathbb{E}[e^{-r(\tau-t)} \mathbf{1}_{\{\tau \leq T_n\}} | S_t]}{\sum_{i=1}^n e^{-r(T_i-t)} \mathbb{P}(\tau > T_i | S_t)}$$

From Proposition 2.8, we know the value of $(\mathbb{P}(\tau > T_i | S_t)_{1 \leq i \leq n})$. It then remains to compute the following quantity $\mathbb{E}[e^{-r\tau} \mathbf{1}_{\tau \leq t}]$ to be able to price the CDS coupon C . By an integration by parts, we show that

$$\mathbb{E}[e^{-r\tau} \mathbf{1}_{\tau \leq t}] = e^{-rt} \mathbb{P}(\tau \leq t) + r \int_0^t e^{-rs} \mathbb{P}(\tau \leq s) ds \quad (22)$$

Otherwise, one could just obtain this expectation by directly using the density of the first-hitting time of 0 that is provided by the differentiation of the cumulative distribution function :

$$f_\tau(t) = \frac{2r(1-\alpha)\zeta_t^{\frac{1}{2(1-\alpha)}} e^{-\zeta_t}}{\Gamma(\frac{1}{2(1-\alpha)})(e^{2(1-\alpha)rt} - 1)}$$

where ζ_t is defined above.

EDSs are very similar to CDSs except that payouts occur when the stock price falls under a pre-defined level, which is often referred to as a trigger price. The trigger price is generally between 30% and 50% of the equity stock price at the beginning of the contract. Hence, these contracts provide a protection against a credit event happening on the equity market for the buyer. They were initiated by the end of 2003. At that time, it had become difficult in many countries to structure investment-grade credit portfolios with good returns because the CDS spreads were tightening, as reported by Sawyer (2003). Another reason why people have interest in those contracts is because the settlement of the default is directly observed on the stock price. Let us now define τ_L as the first passage time of the stock price process under the level $L < S_0$. Formally, we write $\tau_L = \inf\{t > 0; S_t \leq L\}$. We recall the general valuation formula of an EDS:

$$EDS_t(T_1, T_n; C; R) = -C \sum_{i=1}^n e^{-r(T_i-t)} \mathbb{P}(\tau_L > T_i | S_t) + \mathbb{E}[e^{-r(\tau_L-t)} \mathbf{1}_{\{\tau_L \leq T_n\}} | S_t]$$

where C is the coupon, T_1, \dots, T_n the payment dates and r the risk-free interest rate. Again, by absence of arbitrage, we can find the coupon price, by stating that at the initiation of the contract:

$$EDS_{t=0}(T_1, T_n; C; R) = 0$$

Or equivalently

$$C = \frac{\mathbb{E}[e^{-r(\tau_L-t)} \mathbf{1}_{\{\tau_L \leq T_n\}} | S_t]}{\sum_{i=1}^n e^{-r(T_i-t)} \mathbb{P}(\tau_L > T_i | S_t)}$$

In order to price the coupon C , one needs to evaluate:

$$\mathbb{E}[e^{-r\tau_L} \mathbf{1}_{\{\tau_L \leq t\}}] \quad \text{and} \quad \mathbb{P}(\tau_L \leq t)$$

An integration by parts gives the Laplace transform of $\mathbb{P}(\tau_L \leq t)$ for any $\lambda > 0$

$$\int_0^{+\infty} dt e^{-\lambda t} \mathbb{P}(\tau_L \leq t) = \frac{\mathbb{E}[e^{-\lambda \tau_L}]}{\lambda}$$

Applying Fubini theorem, one observes that

$$\int_0^{+\infty} dt e^{-\lambda t} \mathbb{E}[e^{-r\tau_L} \mathbf{1}_{\{\tau_L \leq t\}}] = \frac{\mathbb{E}[e^{-(r+\lambda)\tau_L}]}{\lambda}$$

Hence using Proposition 2.10, one is able to compute the Laplace transform of the desired quantities necessary to evaluate an EDS. One can then use numerical techniques (see Abate and Whitt (1995) for instance) to inverse the Laplace transform in order to evaluate prices.

4 Stochastic Volatility for CEV Processes

4.1 A Zero Correlation Pricing Framework

Impact of a Stochastic Time Change

Due to the very important dependency between the probability of default, the level of volatility and the skewness, we were naturally brought to consider extensions of the CEV model that could relax the high correlation between these three effects. More precisely, in a CEV model, if one first calibrates the implied at-the-money volatility, then either the skewness or the CDS will be calibrated on adjusting the elasticity parameter. Hence, to be able to get some freedom on the volatility surface, a possible extension is to introduce a stochastic volatility in the CEV model instead of a constant volatility. A CEV diffusion with a stochastic volatility is actually just a power of a squared Bessel Process with a stochastic time change instead of having a deterministic one like in Proposition 2.3.

Another extension is to consider a power of a Bessel Process time changed by an independent increasing process. More precisely, one writes the following process for the stock price:

$$\begin{aligned} S_t &= e^{rt} BESQ_{(\delta, x)}^{1-\frac{\delta}{2}}(\xi_t) \quad \text{if} \quad t < \tau. \\ S_t &= 0 \quad \text{if} \quad t \geq \tau. \end{aligned} \tag{23}$$

where $x = S_0^{\frac{2}{2-\delta}}$, $\tau = T_0(S) = \inf\{t > 0, S_t = 0\} = \xi^{-1}(T_0(BESQ))$ and ξ_t is an strictly increasing continuous integrable process independent from the squared Bessel process. Subordinating a continuous process by an independent Lévy process is an idea that goes back to Clark (1973). Stochastic time changes are somehow equivalent to adding a stochastic volatility in stock price diffusions. The basic intuition underlying this approach could be foreseen through the scaling property of the Brownian motion, or through Dambis (1965), Dubins and Schwarz (1965) (DDS) theorem or even its extension to semimartingales by Monroe (1978). More recently, Carr et al. (2003) generated uncertainty by speeding up or slowing down the rate at which time passes with a Lévy process. Our approach differs from the one done in the Lévy processes literature for mathematical finance: We are not considering the exponential of a time

changed Lévy process but a power of a time changed Bessel process. Thanks to Lamperti representation (1972), this means that we are considering a time changed geometric Brownian motion B . More precisely, it is known that

$$R_t = \exp(B_{C_t} + \nu C_t) \quad \text{and} \quad C_t = \int_0^t \frac{ds}{R_s^2}$$

where $(R_t, t \geq 0)$ is a Bessel process of dimension $\delta = 2(1 + \nu)$ starting from $a \neq 0$. Hence the time change considered in the stock price is

$$Y_t = \int_0^{\xi_t} \frac{ds}{R_s^2}$$

and the stock price process as defined in equation (23) can be identified as follows:

$$S_t = e^{rt} \exp\left(-2\nu B_{Y_t} - \frac{(2\nu)^2}{2} Y_t\right)$$

As a consequence, we have now proposed a new class of time changes where analytical computations are possible thanks to a good knowledge of Bessel processes.

For the absence of arbitrage property, there must exist a probability under which all the actualized stock prices are martingales. A very simple property on martingales is that a process M_t is a martingale if and only if for every bounded stopping time T , $\mathbb{E}[M_T] = \mathbb{E}[M_0]$. Nonetheless, this result is not very convenient. Let us state and give a straightforward proof of the martingality of the stock price process

Proposition 4.1 *Consider $M_t = BESQ_{(\delta, x)}^{1-\frac{\delta}{2}}(\xi_{t \wedge \tau})$ where following the previous hypotheses ξ_t is a strictly increasing continuous integrable process independent from $BESQ$, τ is the $(M_t, t \geq 0)$ first hitting time of 0 and $BESQ_{(\delta, x)}$ is a squared Bessel process of dimension δ starting from $x \neq 0$, then $(M_t, t \geq 0)$ is a true martingale.*

Proof : Let us define $\mathcal{R}_t = \sigma(R_s; s \leq t)$. We then naturally write the canonical filtrations $\mathcal{R}_{\xi_t} = \sigma(R_{\xi_s}; s \leq t)$ and $\Xi_t = \sigma(\xi_s; s \leq t)$. For any bounded functional F , we want to compute

$$\mathbb{E}[F(R_{\xi_u}; u \leq s)(R_{\xi_t}^{2-\delta} - R_{\xi_s}^{2-\delta})]$$

Since ξ is integrable and independent from R , we obtain by using Fubini theorem

$$\begin{aligned} \mathbb{E}[F(R_{\xi_u}; u \leq s)(R_{\xi_t}^{2-\delta} - R_{\xi_s}^{2-\delta})] &= \mathbb{E}\left[\mathbb{E}[F(R_{\xi_u}; u \leq s)(R_{\xi_t}^{2-\delta} - R_{\xi_s}^{2-\delta}) | \Xi_t]\right] \\ &= \int \mathbb{P}_{\Xi_t}(da) \mathbb{E}[F(R_{a(u)}; u \leq s)(R_{a(t)}^{2-\delta} - R_{a(s)}^{2-\delta})] \end{aligned}$$

The latest quantity is null by Theorem 2.11 and we have then shown that for $s \leq t < \tau$

$$R_{\xi_s}^{2-\delta} = \mathbb{E}[R_{\xi_t}^{2-\delta} | \mathcal{R}_{\xi_s}]$$

which is the announced result. ■

Pricing Vanilla Options

One can find closed-form formulas for the call and put options prices. Let us define the two following quantities $C_0(x, \delta, K, T; S_0)$ and $P_0(x, \delta, K, T; S_0)$:

$$\begin{aligned} C_0(x, \delta, K, T; S_0) &= S_0 Q\left(\frac{(Ke^{-rT})^{\frac{2}{2-\delta}}}{x}, 4 - \delta, \frac{S_0^{\frac{2}{2-\delta}}}{x}\right) \\ &\quad - Ke^{-rT} \left(1 - Q\left(\frac{S_0^{\frac{2}{2-\delta}}}{x}, 2 - \delta, \frac{(Ke^{-rT})^{\frac{2}{2-\delta}}}{x}\right)\right) \\ P_0(x, \delta, K, T; S_0) &= Ke^{-rT} Q\left(\frac{S_0^{\frac{2}{2-\delta}}}{x}, 2 - \delta, \frac{(Ke^{-rT})^{\frac{2}{2-\delta}}}{x}\right) \\ &\quad - S_0 \left(1 - Q\left(\frac{(Ke^{-rT})^{\frac{2}{2-\delta}}}{x}, 4 - \delta, \frac{S_0^{\frac{2}{2-\delta}}}{x}\right)\right) \end{aligned}$$

From there, one may obtain the option prices under the new general framework.

Proposition 4.2 *If one has the following stock price process:*

$$\begin{aligned} S_t &= e^{rt} BESQ_{(\delta, x)}^{1-\frac{\delta}{2}}(\xi_t) \quad \text{if} \quad t < \tau. \\ S_t &= 0 \quad \text{if} \quad t \geq \tau. \end{aligned}$$

where $x = S_0^{\frac{2}{2-\delta}}$, $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$ and ξ_t is a strictly increasing continuous integrable process independent from $BESQ$ whose probability measure is $\mu_{\xi_t}(dx)$, then:

$$\begin{aligned} C_0 &= \int_{\mathbb{R}_+} C_0(x, \delta, K, T; S_0) \mu_{\xi_T}(dx) \\ P_0 &= \int_{\mathbb{R}_+} P_0(x, \delta, K, T; S_0) \mu_{\xi_T}(dx) \end{aligned}$$

Proof : Let us prove this result for the call option price, a similar result may be obtained for the put price. One has:

$$\begin{aligned} C_0 &= e^{-rT} \mathbb{E}[(S_T - K)_+] \\ &= e^{-rT} \mathbb{E}(\mathbb{E}[(S_T - K)_+ | \sigma(\xi_s; s \leq T)]) \\ &= \mathbb{E}[C_0(\xi_T, \delta, K, T; S_0)] \end{aligned}$$

■

Computing the Default

Having the integrability of the change of time and knowing its density, one could find a closed-form formula for the probability of default $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$ where $S_t = e^{rt} BESQ_{(\delta, x)}^{1-\frac{\delta}{2}}(\xi_t)$. Let us now compute the probability of default the proof of which is left to the reader:

Proposition 4.3 *If one considers a stock price process defined as follows:*

$$S_t = e^{rt} BESQ_{(\delta, x)}^{1-\frac{\delta}{2}}(\xi_t)$$

then the probability of default $\tau = \inf\{t > 0, S_t = 0\}$ is given by

$$\mathbb{P}(\tau \leq T) = \mathbb{E} \left[G \left(1 - \frac{\delta}{2}, \frac{S_0^{\frac{2}{2-\delta}}}{\xi_T} \right) \right]$$

where G is the complementary Gamma function.

4.2 CESV Models

Stochastic volatility models were used in a Black and Scholes (1973) and Merton (1973) framework mainly to capture skewness and kurtosis effects, or in terms of implied volatility skew and smile. In a Constant Elasticity of Variance framework, one would use stochastic volatility not to capture the leverage effect which partly already exists due to the elasticity parameter but to obtain environments for instance of low volatilities, high probabilities of default and low skew. Let us consider an integrable jump-diffusion process $(\sigma_t, t \geq 0)$ to model the volatility. We will call those diffusions Constant Elasticity of Stochastic Variance (CESV) for the remainder of the paper. Leblanc (1997) introduced stochastic volatility for CEV processes.

Hence, the class of models under a risk-neutral probability measure proposed is of the following form:

$$\frac{dS_t}{S_t} = rdt + \sigma_t S_t^{\alpha-1} dW_t$$

where σ is assumed to be independent from the Brownian motion driving the stock price returns. Next, within an equity subject to bankruptcy framework, we are going to stop the diffusion when the stock reaches 0 just as in the previous section. As a consequence, our diffusion becomes:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sigma_t S_t^{\alpha-1} dW_t \quad \text{if } t < \tau. \\ S_t &= 0 \quad \text{if } t \geq \tau. \end{aligned}$$

where $\tau = T_0(S) = \inf\{t > 0, S_t = 0\}$.

Before giving any concrete examples, let us show how CESV models can be seen as Bessel processes with a stochastic time change. So first, let us recall elementary results:

Lemma 4.4 *Let R be a time change with $s \mapsto R_s$ continuous, strictly increasing, $R_0 = 0$ and $R_t < \infty$, for each $t \geq 0$, then for any continuous semimartingale X and any caglad (left continuous with right limits) bounded adapted process H , one has:*

$$\int_0^{R_t} H_s dX_s = \int_0^t H_{R_u} dX_{R_u} \quad (24)$$

Proof : The proof can be found in Revuz and Yor (2001). ■

Then, using Lemma 4.4, (DDS) theorem and Itô formula, we obtain that

$$\begin{aligned} S_t &\stackrel{d}{=} e^{rt} BESQ_{(2-1/(1-\alpha), S_0^{2(1-\alpha)})}^{\frac{1}{2(1-\alpha)}}(H_{t \wedge \tau}) \\ \tau &= \inf \{t \geq 0, S_t = 0\} \\ H_t &= (1-\alpha)^2 \int_0^t \sigma_s^2 e^{-2(1-\alpha)rs} ds \end{aligned}$$

H_t is by construction an increasing continuous integrable process.

Hence $(e^{-rt}S_t, t \geq 0)$ is a continuous martingale by Proposition 4.1. All the results of the previous subsection apply and we are able to compute Vanilla option and CDS prices conditionally on the knowledge of the law of H_t . As a result, we showed that a CESV model is in fact a timed-changed power of Bessel process where the subordinator is an integrated time change $H_t = \int_0^t h_s ds$ with a specific rate of time change h_t that is defined by

$$h_t = (1-\alpha)^2 \sigma_t^2 e^{-2(1-\alpha)rt}$$

We now provide two examples of well-known stochastic volatility models where we compute the law of the time change.

Heston Model Let us first consider a CIR (1985) diffusion for the volatility process

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \eta\sigma_t dW_t^\sigma \quad \text{and} \quad \sigma_0^2 = x > 0$$

where κ , θ and η are strictly positive constants and W^σ is a Brownian motion independent from W . In fact we are proposing a variation of the Heston (1993) model by considering $\alpha \neq 1$. We then want to compute the law of

$$H_t = (1-\alpha)^2 \int_0^t \sigma_s^2 e^{-2(1-\alpha)rs} ds$$

More precisely, we will compute its Laplace transform, that is to say, for any $\lambda > 0$

$$\mathbb{E}[e^{-\lambda H_t}]$$

For this purpose, let us use the following result:

Lemma 4.5 *If X a squared Bessel process $BESQ_{(\delta, x)}$ starting from $x \neq 0$ and of dimension δ , then for any function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any $t > 0$: $\int_0^t f(s)ds < \infty$, we have*

$$\mathbb{E}\left[\exp\left(-\int_0^t X_s f(s)ds\right)\right] = \frac{1}{\psi_f'(t)^{\delta/2}} \exp\frac{x}{2}\left(\phi_f'(0) - \frac{\phi_f'(t)}{\psi_f'(t)}\right)$$

where ϕ_f is the unique solution of the Sturm-Liouville equation

$$\phi_f''(s) = 2f(s)\phi_f(s)$$

where $s \in [0; \infty[$, $\phi_f(0) = 1$, ϕ_f is positive and non-increasing and

$$\psi_f(t) = \phi_f(t) \int_0^t \frac{ds}{\phi_f^2(s)}$$

Proof : The proof can be found in Pitman and Yor (1982). ■

By Lemma 2.1, we can see that

$$H_t \stackrel{d}{=} \left(\frac{2(1-\alpha)}{\eta} \right)^2 \int_0^{\frac{\eta^2}{4\kappa}(e^{\kappa t} - 1)} X_u \frac{du}{\left(\frac{4\kappa u}{\eta^2} + 1 \right)^{2\left[\frac{(1-\alpha)r}{\kappa} + 1 \right]}}$$

where X is a $BESQ_{(\frac{4\kappa\theta}{\eta^2}, x)}$. Hence for any $\lambda > 0$,

$$\mathbb{E}[e^{-\lambda H_t}] = \mathbb{E} \left[\exp \left(- \int_0^{l(t)} X_s f_\lambda(s) ds \right) \right]$$

with

$$l(t) = \frac{\eta^2}{4\kappa}(e^{\kappa t} - 1) \quad \text{and} \quad f_\lambda(t) = \lambda \left(\frac{2(1-\alpha)}{\eta} \right)^2 \left(\frac{4\kappa u}{\eta^2} + 1 \right)^{-2\left[\frac{(1-\alpha)r}{\kappa} + 1 \right]} \quad (25)$$

Defining $a = 8((1-\alpha)/\eta)^2$, $b = 4\kappa/\eta^2$ and $n = -2\left(\frac{(1-\alpha)r}{\kappa} + 1\right)$ and using Lemma 4.5 we are brought to the resolution of the following ordinary differential equation

$$\phi''(x) - a\lambda(bx + 1)^n \phi(x) = 0$$

Then under the boundary conditions, one obtains (see Polyanin and Zaitsev (2003)):

$$\begin{aligned} \phi_\lambda(x) &= \sqrt{bx+1} \frac{\frac{\pi}{\sin(\nu\pi)} I_{1/(n+2)} \left(\frac{2\sqrt{a\lambda}}{b(n+2)} (bx+1)^{(n+2)/2} \right)}{I_{-1/(n+2)} \left(\frac{2\sqrt{a\lambda}}{b(n+2)} \right)} \\ &\quad + \sqrt{bx+1} \frac{K_{1/(n+2)} \left(\frac{2\sqrt{a\lambda}}{b(n+2)} (bx+1)^{(n+2)/2} \right)}{I_{-1/(n+2)} \left(\frac{2\sqrt{a\lambda}}{b(n+2)} \right)} \end{aligned} \quad (26)$$

$$\begin{aligned} \psi_\lambda(x) &= C_1 \sqrt{bx+1} I_{1/(n+2)} \left(\frac{2\sqrt{a\lambda}}{b(n+2)} (bx+1)^{(n+2)/2} \right) \\ &\quad + C_2 \sqrt{bx+1} K_{1/(n+2)} \left(\frac{2\sqrt{a\lambda}}{b(n+2)} (bx+1)^{(n+2)/2} \right) \end{aligned} \quad (27)$$

where with using the fact that $I'_\nu(x)K_\nu(x) - I_\nu(x)K'_\nu(x) = 1/x$ one has

$$\begin{aligned} C_1 &= -\frac{b(n+2)}{2a\lambda} K_{1/(n+2)}\left(\frac{2\sqrt{a\lambda}}{b(n+2)}\right) \\ C_2 &= \frac{b(n+2)}{2a\lambda} I_{1/(n+2)}\left(\frac{2\sqrt{a\lambda}}{b(n+2)}\right) \end{aligned}$$

We finally obtain the Laplace transform of H_t for any $\lambda > 0$

$$\mathbb{E}[e^{-\lambda H_t}] = \frac{1}{\psi'_\lambda(l(t))^{\delta/2}} \exp \frac{x}{2} \left(\phi'_\lambda(0) - \frac{\phi'_\lambda(l(t))}{\psi'_\lambda(l(t))} \right)$$

with $\delta = \frac{4\kappa\theta}{\eta^2}$.

A simpler example for the forward contract is provided in Atlan and Leblanc (2005).

Hull and White Model Let us now consider the Hull and White (1987) volatility diffusion that is driven by the following stochastic differential equation:

$$\frac{d\sigma_t^2}{\sigma_t^2} = \theta dt + \eta dW_t^\sigma$$

where θ and η are positive constants and W^σ is a Brownian motion independent from W . Then H may be computed and after some simplifications, we obtain:

$$H_t = \frac{4(1-\alpha)\sigma_0^2}{\eta^2} \int_0^{\frac{\eta^2 t}{4}} ds e^{2(W_s^\sigma + \nu s)} \quad (28)$$

where $\nu = \frac{2}{\eta^2}(\theta - \frac{\eta^2}{2} - 2(1-\alpha)r)$.

If we define $A_t^\nu = \int_0^t \exp 2(B_s + \nu s) ds$ where B is a Brownian motion, we recognize a typical quantity used for the pricing of Asian options with analytical formulae. Thus, we can write

$$H_t = \frac{4(1-\alpha)\sigma_0^2}{\eta^2} A_{\frac{\eta^2 t}{4}}^\nu$$

and obtain its law using Yor (1992), more precisely we have $\forall (u, v) \in \mathbb{R}_+^2$:

$$f_{|A_t^\nu}(u) = \frac{\exp\left(\frac{\pi^2}{2t} - \frac{\nu^2 t}{t} - \frac{1}{2u}\right)}{u^2 \sqrt{2\pi^3 t}} \int_{-\infty}^{+\infty} dx e^{x(\nu+1)} e^{-\frac{e^{2x}}{2u}} \psi_{\frac{e^x}{u}}(t) \quad (29)$$

where:

$$\psi_r(v) = \int_0^\infty dy \exp\left(-\frac{y^2}{2v}\right) e^{-r \cosh(y)} \sinh(y) \sin\left(\frac{\pi y}{v}\right) \quad (30)$$

4.3 Subordinated Bessel Models

Another way to build stochastic volatility models is to make time stochastic. Geman, Madan and Yor (2001) recognize that asset prices may be viewed as Brownian motions subordinated by a random clock. The random clock may be regarded as a cumulative measure of the economic activity as said in Clark (1973) and as estimated in Ané and Geman (2000). The time must be an increasing process, thus it could either be a Lévy subordinator or a time integral of a positive process. In this paper, we only consider the case of a time integral because we need the continuity of the time change in order to compute the first-passage time at 0 to be able to provide analytical formulas for CDS prices. More generally, for the purpose of pricing path-dependent options, one needs the continuity of the time change in order to simulate increments of the time changed Bessel process. Consequently, we study the case of a time change Y_t such as

$$Y_t = \int_0^t y_s ds$$

where the rate of time change $(y_t, t \geq 0)$ is a positive stochastic process. As we have seen in the previous subsection, considering a stochastic volatility $(\sigma_t, t \geq 0)$ in the CEV diffusion is equivalent to the following rate of time change

$$y_t = \frac{\sigma_t^2 e^{\frac{2rt}{\delta-2}}}{(2-\delta)^2}$$

where δ is the dimension of the squared Bessel process. Hence, in order to provide frameworks where one is able to compute the law of the time change, we are going to go directly through different modellings of the rate of time change y_t .

Integrated CIR Time change As a first example, let us consider the case where y_t solves the following diffusion

$$dy_t = \kappa(\theta - y_t)dt + \eta\sqrt{y_t}dW_t^Y$$

where W^Y is independent from the driving Bessel process. The Laplace transform of Y_t is then defined for any $\lambda > 0$ by :

$$\begin{aligned} \mathbb{E}[e^{-\lambda Y_t}] &= e^{\frac{\kappa^2 \theta t}{\eta^2}} \frac{\exp\left(-2\lambda y_0/(\kappa + \gamma \coth(\gamma t/2))\right)}{\left(\cosh(\gamma t/2) + \frac{\kappa}{\gamma} \sinh(\gamma t/2)\right)^{2\kappa\theta/\eta^2}} \\ \gamma &= \sqrt{\kappa^2 + 2\eta^2\lambda} \end{aligned}$$

Integrated Ornstein-Uhlenbeck Time Change We now assume the rate of time change to be the solution of the following SDE

$$dy_t = -\lambda y_t dt + dz_t$$

where $(z_t; t \geq 0)$ is a Lévy subordinator. Let ψ_z denote the log characteristic function of the subordinator z_t , then

$$\mathbb{E}[e^{iaY_t}] = \exp\left(ia y_0 \frac{1 - e^{-\lambda t}}{\lambda}\right) \exp\left(\int_0^{a \frac{1 - e^{-\lambda t}}{\lambda}} \frac{\psi_z(x)}{a - \lambda x} dx\right) \quad (31)$$

Then we can compute the characteristic function of Y_t for different subordinators and we present here three examples that one can find in Carr et al. (2003) for which we recall below the characteristic functions:

a) For a process with Poisson arrival rate ν of positive jumps exponentially distributed with mean μ , we have a Lévy density that is

$$k_z(x) = \frac{\nu}{\mu} e^{-\frac{x}{\mu}} \mathbf{1}_{\{x>0\}}$$

and a log characteristic function

$$\psi_z(x) = \frac{ix\nu\mu}{1 - ix\mu}$$

then we obtain

$$\int \frac{\psi_z(x)}{a - \lambda x} dx = \log\left(\left(x + \frac{i}{\mu}\right)^{\frac{\nu}{\lambda - i\mu a}} (a - \lambda x)^{\frac{\nu a \mu}{\lambda a \mu + i\lambda}}\right) \quad (32)$$

b) Let us consider the first time a Brownian motion with drift ν reaches 1. It is well known that this passage time follows the so-called Inverse Gaussian law which Lévy density and log characteristic function are respectively

$$\begin{aligned} k_z(x) &= \frac{e^{-\frac{\nu^2 x}{2}}}{\sqrt{2\pi x^3}} \mathbf{1}_{\{x>0\}} \\ \psi_z(x) &= \nu - \sqrt{\nu^2 - 2ix} \end{aligned}$$

and we then get

$$\begin{aligned} \int \frac{\psi_z(x)}{a - \lambda x} dx &= \frac{2\sqrt{\nu^2 - 2ix}}{\lambda} + \frac{2\sqrt{\nu^2 \lambda - 2ia}}{\lambda^{3/2}} \operatorname{arctanh}\left(\sqrt{\frac{\lambda(\nu^2 - 2ix)}{\nu^2 \lambda - 2ia}}\right) \\ &\quad - \frac{\nu \log(a - \lambda x)}{\lambda} \end{aligned}$$

c) Finally, recall the Stationary Inverse Gaussian case which Lévy density and log characteristic function are

$$\begin{aligned} k_z(x) &= \frac{(1 + \nu^2 x) e^{-\frac{\nu^2 x}{2}}}{2\sqrt{2\pi x^3}} \mathbf{1}_{\{x>0\}} \\ \psi_z(x) &= \frac{i\nu}{\sqrt{\nu^2 - 2ix}} \end{aligned}$$

From these definitions, we obtain

$$\int \frac{\psi_z(x)}{a - \lambda x} dx = \frac{\sqrt{\nu^2 - 2ix}}{\lambda} - \frac{2ia}{\lambda^{3/2} \sqrt{\nu^2 \lambda - 2ia}} \operatorname{arctanh}\left(\sqrt{\frac{\lambda(\nu^2 - 2ix)}{\nu^2 \lambda - 2ia}}\right)$$

5 Correlation Adjustment

5.1 Introducing some Correlation

We propose a time-changed Bessel process as in the previous section with some leverage in order to get more independence between skewness and credit spreads, with respect to which we add a term that contains a negative correlation (equal to ρ) component between the stock return and the volatility. Hence, let us consider z_t a $\sigma(h_s, s \leq t)$ adapted positive integrable process such as

$$\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}$$

is a martingale and a general integrated time change $H_t = \int_0^t h_s ds$ such as $\mathbb{E}(H_t) < \infty$ then, we can define the stock price process as follows

$$\begin{aligned} S_t &= e^{rt} BESQ_{H_t \wedge \tau}^{2-\delta} \frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]} \\ \tau &= \inf\{t > 0; S_t = 0\} \end{aligned}$$

where $BESQ$ is a squared Bessel process of dimension $\delta < 2$ starting from $S_0^{1/(2-\delta)}$.

Let us first show that the process $(e^{-rt} S_t; t \geq 0)$ hence defined is a martingale. We know from Proposition 4.1 that $BESQ_{H_t \wedge \tau}^{2-\delta}$ is a martingale. Now because of the independence of the processes z and $BESQ$

$$\langle BESQ_{H_t \wedge \tau}^{2-\delta}, \frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]} \rangle_t = 0$$

which ensures that $(e^{-rt} S_t; t \geq 0)$ is a local martingale. Let us show that it is actually a true martingale. For this purpose, let us recall some results:

Definition 5.1 *A real valued process X is of class DL if for every $a > 0$, the family of random variables $X_T \mathbf{1}_{\{T < a\}}$ is uniformly integrable for all stopping times.*

We now state the following property:

Proposition 5.2 *Let M_t be a local martingale such that $\mathbb{E}|M_0| < \infty$ and such that its negative part belongs to class DL. Then its negative part is a supermartingale. M_t is a martingale if and only if $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for all $t > 0$.*

Proof : The proof may be found in Elworthy, Li and Yor (1999). ■

All the financial assets being positive, one may use a simpler result than the previous property the proof of which is left to the reader:

Corollary 5.3 *Let M_t be a positive local martingale such that $\mathbb{E}[M_0] < \infty$. Then M_t is a supermartingale and it is a martingale if and only if $\mathbb{E}[M_t] = \mathbb{E}[M_0]$ for all $t > 0$.*

Consequently to prove that the actualized stock price process is a martingale with regards to the filtration $\mathcal{F}_t = \mathcal{R}_{H_t} \vee \sigma(h_s; s \leq t)$, we just need to show that for any $t > 0$

$$\mathbb{E}[e^{-rt}S_t] = S_0$$

which is the case since

$$\begin{aligned} \mathbb{E}[e^{-rt}S_t] &= \mathbb{E}[BESQ_{H_t \wedge \tau}^{2-\delta} \frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}] = \mathbb{E}\left[\mathbb{E}[BESQ_{H_t \wedge \tau}^{2-\delta} \frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]} | \sigma(h_s; s \leq t)]\right] \\ &= \mathbb{E}\left[\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]} \mathbb{E}[BESQ_{H_t \wedge \tau}^{2-\delta} | \sigma(h_s; s \leq t)]\right] = \mathbb{E}\left[\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]} S_0\right] = S_0 \end{aligned}$$

5.2 Pricing Credit and Equity Derivatives

The computation of the probability of default is immediate from Proposition 4.3 because

$$\tau = \inf\{t \geq 0; S_t = 0\} = \inf\{t \geq 0; BESQ_{H_t} = 0\}$$

and then for any $T > 0$

$$\mathbb{P}(\tau \leq T) = \mathbb{E}\left[G\left(1 - \frac{\delta}{2}, \frac{S_0^{\frac{2}{2-\delta}}}{H_T}\right)\right]$$

where G is the complementary Gamma function.

Let us compute the European vanilla option prices. For this purpose, we define $C_0^{(\rho)}(x, y, \delta, K, T; S_0)$ and $P_0^{(\rho)}(x, y, \delta, K, T; S_0)$:

$$\begin{aligned} C_0^{(\rho)}(x, y, \delta, K, T; S_0) &= S_0 \frac{e^{\rho z_T}}{\mathbb{E}[e^{\rho z_T}]} Q\left(\frac{(K e^{-(rT+\rho y)} \mathbb{E}[e^{\rho z_T}])^{\frac{2}{2-\delta}}}{x}, 4 - \delta, \frac{S_0^{\frac{2}{2-\delta}}}{x}\right) \\ &\quad - K e^{-rT} \left(1 - Q\left(\frac{S_0^{\frac{2}{2-\delta}}}{x}, 2 - \delta, \frac{(K e^{-(rT+\rho y)} \mathbb{E}[e^{\rho z_T}])^{\frac{2}{2-\delta}}}{x}\right)\right) \\ P_0^{(\rho)}(x, y, \delta, K, T; S_0) &= K e^{-rT} Q\left(\frac{S_0^{\frac{2}{2-\delta}}}{x}, 2 - \delta, \frac{(K e^{-(rT+\rho y)} \mathbb{E}[e^{\rho z_T}])^{\frac{2}{2-\delta}}}{x}\right) \\ &\quad - S_0 \frac{e^{\rho z_T}}{\mathbb{E}[e^{\rho z_T}]} \left(1 - Q\left(\frac{(K e^{-(rT+\rho y)} \mathbb{E}[e^{\rho z_T}])^{\frac{2}{2-\delta}}}{x}, 4 - \delta, \frac{S_0^{\frac{2}{2-\delta}}}{x}\right)\right) \end{aligned}$$

Then, the knowledge of the joint law μ_{H_t, z_t} for any $t > 0$ enables us to compute the option prices as in the previous section:

$$\begin{aligned} C_0 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} C_0^{(\rho)}(x, y, \delta, K, T; S_0) \mu_{H_t, z_t}(dx, dy) \\ P_0 &= \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} P_0^{(\rho)}(x, y, \delta, K, T; S_0) \mu_{H_t, z_t}(dx, dy) \end{aligned}$$

5.3 Examples

Let us go through most of the time changes presented previously and see how we can obtain the joint law of the couple (H_t, z_t) .

Integrated CIR Time change Let us consider the following dynamics

$$dh_t = \kappa(\theta - h_t)dt + \eta\sqrt{h_t}dW_t^H$$

where W^H and $BESQ$ are independent and the stability condition $\frac{2\kappa\theta}{\eta^2} > 1$ is satisfied. Let us take

$$z_t = h_t + (\kappa - \frac{\rho\eta^2}{2})H_t$$

or equivalently

$$\rho z_t = \rho(h_0 + \kappa\theta t) + \rho\eta \int_0^t \sqrt{h_s}dW_s^H - \frac{\rho\eta^2}{2} \int_0^t h_s ds$$

Hence, it is obvious that

$$\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}$$

is a local martingale and it is known that it is a martingale as one may check using the Laplace transform below, that

$$\mathbb{E}[\exp(\rho\eta \int_0^t \sqrt{h_s}dW_s^H - \frac{\rho\eta^2}{2} \int_0^t h_s ds)] = 1$$

In order to compute credit and equity derivatives prices, we then compute for any positive λ, μ the Laplace transform of

$$\mathbb{E}[e^{-\lambda H_t - \mu h_t}]$$

It is well known (see Karatzas and Shreve (1991) or Lamberton and Lapeyre (1995)) that

$$\begin{aligned} \mathbb{E}[e^{-\lambda H_t - \mu h_t}] &= \frac{e^{\frac{\kappa^2 \theta t}{\eta^2}}}{\left(\cosh(\gamma t/2) + \frac{\kappa + \mu\eta^2}{\gamma} \sinh(\gamma t/2) \right)^{2\kappa\theta/\eta^2}} \exp(-h_0 B(t, \lambda, \mu)) \\ B(t, \lambda, \mu) &= \frac{\mu(\gamma \cosh(\frac{\gamma t}{2}) - \kappa \sinh(\frac{\gamma t}{2})) + 2\lambda \sinh(\frac{\gamma t}{2})}{\gamma \cosh(\frac{\gamma t}{2}) + (\kappa + \lambda\eta^2) \sinh(\frac{\gamma t}{2})} \\ \gamma &= \sqrt{\kappa^2 + 2\eta^2 \lambda} \end{aligned}$$

Heston CESV with correlation In the same class of models, let us now construct z in terms of the solution of the following stochastic differential equation

$$d\sigma_t^2 = \kappa(\theta - \sigma_t^2)dt + \eta\sigma_t dW_t^H \quad \text{and} \quad \sigma_0^2 = x.$$

First, h is defined by

$$h_t = \frac{\sigma_t^2 e^{2rt/(\delta-2)}}{(2-\delta)^2}$$

Then, following the same method as in the integrated CIR time change case, we choose z as:

$$z_t = h_t + \left(\kappa - \frac{2r}{\delta-2} - \frac{\rho\eta^2}{2}\right)H_t$$

Consequently, $\frac{e^{\rho z_t}}{\mathbb{E}[e^{\rho z_t}]}$ is a martingale.

Hence, it remains to evaluate for any positive t the Laplace transform of (H_t, z_t) , that is to say for any positive λ, μ

$$\mathbb{E}[e^{-\lambda H_t - \mu h_t}]$$

In order to compute the above quantity, we use the following result which extends Lemma 4.6 that one can find in Pitman and Yor (1982).

Lemma 5.4 *If X a squared Bessel process $BESQ_{(\delta, x)}$ starting from $x \neq 0$ and of dimension δ , then for any functions f and $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for any $t > 0$: $\int_0^t f(s)ds < \infty$, we have*

$$\begin{aligned} \mathbb{E}\left[\exp\left(-\int_0^t X_s f(s)ds - g(t)X_t\right)\right] &= \frac{1}{(\psi'_f(t) + 2g(t)\psi_f(t))^{\delta/2}} \times \\ &\quad \exp\frac{x}{2}\left(\phi'_f(0) - \frac{\phi'_f(t) + 2g(t)\phi_f(t)}{\psi'_f(t) + 2g(t)\psi_f(t)}\right) \end{aligned}$$

where ϕ_f is the unique solution of the Sturm-Liouville equation

$$\phi_f''(s) = 2f(s)\phi_f(s)$$

where $s \in [0; \infty[$, $\phi_f(0) = 1$, ϕ_f is positive and non-increasing and

$$\psi_f(t) = \phi_f(t) \int_0^t \frac{ds}{\phi_f^2(s)}$$

Taking

$$g(t) = \frac{\mu}{(2-\delta)^2} \left(\frac{\eta^2}{4\kappa t + \eta^2} \right)^{1 + \frac{2r}{\kappa(2-\delta)}}$$

in the above Lemma, we obtain that for any positive λ, μ

$$\begin{aligned} \mathbb{E}[e^{-\lambda H_t - \mu h_t}] &= \frac{1}{(\psi'_\lambda(l(t)) + 2\frac{\mu}{(2-\delta)^2} e^{-(\kappa + \frac{2r}{2-\delta})t} \psi_\lambda(l(t)))^{\delta/2}} \times \\ &\quad \exp\frac{x}{2}\left(\phi'_\lambda(0) - \frac{\phi'_\lambda(l(t)) + 2\frac{\mu}{(2-\delta)^2} e^{-(\kappa + \frac{2r}{2-\delta})t} \phi_\lambda(l(t))}{\psi'_\lambda(l(t)) + 2\frac{\mu}{(2-\delta)^2} e^{-(\kappa + \frac{2r}{2-\delta})t} \psi_\lambda(l(t))}\right) \end{aligned}$$

where noting $\alpha = \frac{\delta-1}{\delta-2}$, the functions ϕ_λ , ψ_λ and l are defined respectively in (25), (26) and (27).

Integrated Ornstein-Uhlenbeck Time Change We consider the stochastic time change $H_t = \int_0^t h_s ds$ and assume that $(h_t; t \geq 0)$ is given by

$$dh_t = -\lambda h_t dt + dz_t$$

where $(z_t; t \geq 0)$ is a Lévy subordinator. Carr et al. (2003) compute the characteristic function $\Phi(t, a, b)$ of (H_t, z_t) for any $t > 0$ and it is given by

$$\mathbb{E}[e^{iaH_t + ibz_t}] = \exp\left(iah_0 \frac{1 - e^{-\lambda t}}{\lambda}\right) \exp\left(\int_b^{b+a \frac{1-e^{-\lambda t}}{\lambda}} \frac{\psi_z(x)}{a + \lambda b - \lambda x} dx\right) \quad (33)$$

for any $(a, b) \in \mathbb{R}_+^2$ where ψ_z is the log characteristic function of the subordinator. Let us first notice that

$$\mathbb{E}[e^{\rho z_t}] = \exp(t\psi_z(-i\rho))$$

We quickly recall the computations of $\Phi(t, a, b)$ for different subordinators:

a) For a process with Poisson arrival rate ν of positive jumps exponentially distributed with mean μ , we obtain

$$\int \frac{\psi_z(x)}{a + \lambda b - \lambda x} dx = \log\left(\left(x + \frac{i}{\mu}\right)^{\frac{\nu}{\lambda - i\mu(a + \lambda b)}} ((a + \lambda b) - \lambda x)^{\frac{\nu(a + \lambda b)\mu}{\lambda(a + \lambda b)\mu + i\lambda}}\right)$$

b) For an Inverse Gaussian subordinator of parameter ν , we have

$$\begin{aligned} \int \frac{\psi_z(x)}{a + \lambda b - \lambda x} dx &= \frac{2\sqrt{\nu^2 - 2ix}}{\lambda} \\ &+ \frac{2\sqrt{\nu^2\lambda - 2i(a + \lambda b)}}{\lambda^{3/2}} \operatorname{arctanh}\left(\sqrt{\frac{\lambda(\nu^2 - 2ix)}{\nu^2\lambda - 2i(a + \lambda b)}}\right) \\ &- \frac{\nu \log((a + \lambda b) - \lambda x)}{\lambda} \end{aligned}$$

c) For the Stationary Inverse Gaussian of parameter ν , we write

$$\begin{aligned} \int \frac{\psi_z(x)}{a + \lambda b - \lambda x} dx &= \frac{\sqrt{\nu^2 - 2ix}}{\lambda} \\ &- \frac{2i(a + \lambda b)}{\lambda^{3/2}\sqrt{\nu^2\lambda - 2i(a + \lambda b)}} \operatorname{arctanh}\left(\sqrt{\frac{\lambda(\nu^2 - 2ix)}{\nu^2\lambda - 2i(a + \lambda b)}}\right) \end{aligned}$$

6 Conclusion

Twelve continuous stochastic stock price models were built in this paper for equity-credit modelling purposes, all derived from the Constant Elasticity of Variance model, and as a consequence from Bessel processes. They all exploit the ability of Bessel processes to be positive, for those of dimension lower than 2 to reach 0 and for a certain power of a given Bessel process to be a martingale. We first propose to add a stochastic volatility diffusion to the CEV model, then more generally to stochastically time change a Bessel process in order to obtain a stochastic volatility effect, motivated by known arguments that go back to Clark (1973). Next, in order to add some correlation between the stock price process and the stochastic volatility, we extend our framework by multiplying the Bessel process by exponentials of the volatility and correcting it by its mean in accordance with arbitrage considerations to obtain martingale models that are martingales with respect to the joint filtration of the time-changed Bessel process and the stochastic time change itself. Hence, among the different models proposed based on the CEV with default model, there were first the Constant Elasticity of Stochastic Variance ones (CESV) taking a Hull and White (1987) stochastic volatility as well as a Heston (1993) one. We then proposed integrated time change models, by considering an integrated CIR time change and an Integrated Ornstein-Uhlenbeck time change (see Carr et al. (2003)) with different subordinators for the process driving the diffusion. We finally added correlation between stock price returns and volatilities to the models presented previously and provided quasi-analytical formulas for option and CDS prices for all of them. Let us note that we discussed the true and local strict martingale properties of CEV processes, that we naturally extended to the time change framework.

The models presented and discussed in this paper are not specifically designed to cope just with Equity-Credit frameworks but they also can be used for instance for FX-rates hybrid modelling by specifying stochastic interest rates. We can also note that a Poisson jump to default process can be added to the CEV-like framework in order to deal with credit spreads for short-term maturities. Campi, Polbennikov and Sbuelz (2005) and Carr and Linetsky (2005) precisely considered a CEV model with deterministic volatilities and hazard rates. The latest paper can easily be generalized to fit in our time-changed Bessel frameworks. Since our goal was to concentrate on continuous diffusions, we leave the addition of a jump to default for further research.

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